Proseminar I protocol

Alexander Klink

29.10.1999

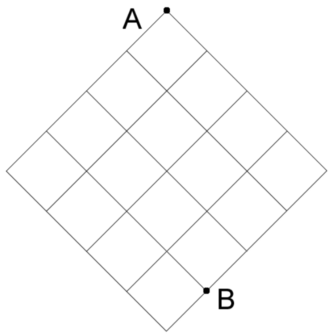

Figure 1: The task

Exercises

1

1 a) What is the shortest route from A to B?
1 b) How many "shortest" routes are there from A to B?
1 c) Generalize your results.

2

2 a) Multiply
$(a_1 + b_1)(a_2 + b_2)$
$(a_1 + b_1)(a_2 + b_2)(a_3 + b_3)$
$(a_1 + b_1)(a_2 + b_2)(a_3 + b_3)(a_4 + b_4)$
and describe the patterns.
2 b) Drop the subscripts to find formulae for

$(a+b)^2$
$(a+b)^3$
$(a+b)^4$

2 c) Explain why the coefficient of a^3b in $(a+b)^4$ is $\binom{4}{3} = \binom{4}{1}$
2 d) Explain why the coefficient of $a^{17}b^6$ in $(a+b)^{23}$ is what it is.

3

3 a) How many subsets does an n-element set have?
3 b) Find the sum of the nth row of Pascal's triangle.

Answers

1 a)

The shortest route from A to B is 7 blocks long. It is the shortest one, because to get the shortest way, we only have to move south (i.e. either south-east or south-west). Thus we always have to move 4 blocks south-east and 3 blocks south-west.

1 b)

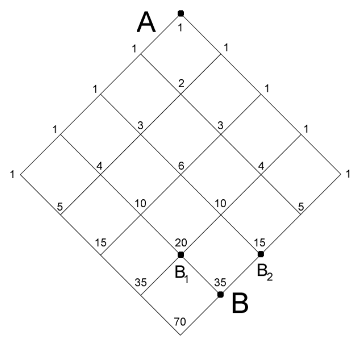

Figure 2: Sketch of a solution

There are 35 different shortest routes from A to B, as you can only get to B by passing either B_1 or B_2. As there are 20 different shortest ways to get to B_1 and 15 ways to get to B_2, we end up with 35 different ways to arrive at B. The values for B_1 and B_2 are computed in a similar way, which leads us to recognizing the figure as a part of Pascal's triangle (cf. fig. 2).

$$\begin{array}{ccccccccccc|l}
 & & & & & 1 & & & & & & \text{0th row} \\
 & & & & 1 & & 1 & & & & & \text{1st row} \\
 & & & 1 & & 2 & & 1 & & & & \text{2nd row} \\
 & & 1 & & 3 & & 3 & & 1 & & & \text{3rd row} \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & \text{4th row} \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 & \text{5th row}
\end{array}$$

$$\begin{array}{ccccccccccc|l}
 & & & & & \binom{0}{0} & & & & & & \text{0th row} \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \text{1st row} \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \text{2nd row} \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \text{3rd row} \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \text{4th row} \\
\binom{5}{0} & & \binom{5}{1} & & \binom{4}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \text{5th row}
\end{array}$$

Figure 3: Pascal's triangle with integers and binomial coefficients

1 c)

If you look at the figure in the way described above, i.e. by recognizing the pattern of Pascal's triangle (cf. fig. 3), one can see that if you move n blocks to the south-east and m blocks to the south-west, the shortest routes are all $n+m$ blocks long. You have $\binom{n+m}{n} = \binom{n+m}{m}$ different routes to get there. One can explain this, because you have to decide on each corner whether to move south-east or south-west. You decide n times to move to the south-east and m times to move to the south-west. So you can ask yourself the question: "How many different ways are there to decide n times to move to the south-east in a $n+m$ journey?". This question evidently leads us to the solution of $\binom{n+m}{n}$. By the same argument, you can use $\binom{n+m}{m}$ as well.

2 a)

$(a_1 + b_1)(a_2 + b_2) = a_1a_2 + a_1b_2 + b_1a_2 + b_1b_2$

$(a_1 + b_1)(a_2 + b_2)(a_3 + b_3) = a_1a_2a_3 + a_1a_2b_3 + a_1b_2a_3 + a_1b_2b_3 + b_1a_2a_3 + b_1a_2b_3 + b_1b_2a_3 + b_1b_2b_3$
$(a_1+b_1)(a_2+b_2)(a_3+b_3)(a_4+b_4) = a_1a_2a_3a_4+a_1a_2b_3a_4+a_1b_2a_3a_4+a_1b_2b_3a_4+ b_1a_2a_3a_4 + b_1a_2b_3a_4 + b_1b_2a_3a_4 + b_1b_2b_3a_4 + a_1a_2a_3b_4 + a_1a_2b_3b_4 + a_1b_2a_3n_4 + a_1b_2b_3n_4 + b_1a_2a_3b_4 + b_1a_2b_3b_4 + b_1b_2a_3b_4 + b_1b_2b_3b_4$
In these terms, one can already recognize the pattern of Pascal's triangle. For example, in the second equation we have one pair that has 3 a's in it ($a_1a_2a_3$),

as well as one pair with 3 b's in it ($b_1 b_2 b_3$). As well, we have got 3 terms with 2 a's in it and 3 terms with 2 b's in it.
All in all, there are 2^n terms, where n is the maximum index of the variables, because each term in braces gives us the possibility to choose either a variable with an a or one with a b in it to multiply the rest with.

2 b)

As already noted above, by dropping the subscripts, we can combine the terms as follows:
$(a+b)^2 = a^2 + 2ab + b^2$
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

2 c)

To explain why the coefficient of a^3b in $(a+b)^4$ equals $\binom{4}{3} = \binom{4}{1}$, we use the subscripts again:

a_1	a_2	a_3	b_4
a_1	a_2	b_3	a_4
a_1	b_2	a_3	a_4
b_1	a_2	a_3	a_4

By this, you can see that there are $\binom{4}{3}$ ways to combine 3 a's in a term with 4 elements. If you look at it from a different angle, you can also consider how many different ways there are to combine 1 b in this term, i.e $\binom{4}{1}$.

2 d)

The coefficient of $a^{17}b^6$ in $(a+b)^{23}$ is $\binom{23}{17}$ or $\binom{23}{6}$.
The argument is similar to the one used in 2 c), there are $\binom{23}{17}$ different ways to combine 17 a's in a term with 23 elements or $\binom{23}{6}$ different ways to combine 6 b's in the same term.

3 a)

A n-element set has 2^n subsets.
Explanation: If you have a set and want to create a subset of it, you consider every element and decide for each element whether to "keep" it or to "throw it out". Therefore, you have 2 different chances for every element. Because you have n elements in the set, you have to multiply 2 with itself n times. I.e. you can create 2^n different subsets from it.

$$\begin{array}{ccccccccccccccc}
 & & & & & 1 & & & & & & = & 1 \\
 & & & & 1 & + & 1 & & & & & = & 2 \\
 & & & 1 & + & 2 & + & 1 & & & & = & 4 \\
 & & 1 & + & 3 & + & 3 & + & 1 & & & = & 8 \\
 & 1 & + & 4 & + & 6 & + & 4 & + & 1 & & = & 16 \\
1 & + & 5 & + & 10 & + & 10 & + & 5 & + & 1 & = & 32 \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & & \\
0 & & 1 & & 2 & & 3 & & 4 & & 5 & & \text{elements}
\end{array}$$

Figure 4: The sum of the elements in Pascal's triangle

3 b)

For exactly the same reasons as mentioned above, the sum of the nth row of Pascal's triangle equals 2^n. If you look at fig. 4, you can see that the numbers in Pascal's triangle correspond directly to the binomial coefficients, which also tell you how many different ways there are to create different subsets from a 5-element list. I.e. the first column in the last row says that there is one set with 0 elements (i.e. the empty set), 5 sets with one element, 10 sets with 2 elements, 10 sets with 3 elements, 5 sets with 4 elements and exactly one set with 5 elements.
You can see that this corresponds exactly to question 3 a).

Short "lecture" on $\binom{n}{k}$

$\binom{n}{k}$ (speak: n choose k) equals the number of ways you have to choose a subset of k objects from a set with n objects.
The order in lists does *not* matter: $\{c, b, a\} = \{a, b, c\}$
$\binom{18}{3} = \frac{18 \cdot 17 \cdot 16}{3 \cdot 2 \cdot 1}$. Here, the numerator denotes the number of ways to choose 3 elements *in order*. The denominator denotes the number of ways the 3 chosen items may be ordered.
To simplify this expression by writing it with factorials, we extend the fraction: $\binom{18}{3} = \frac{18 \cdot 17 \cdot 16 \cdot (15 \cdot 14 \cdot 13 \cdot \dots \cdot 3 \cdot 2 \cdot 1)}{3 \cdot 2 \cdot 1 \cdot (15 \cdot 14 \cdot 13 \cdot \dots \cdot 3 \cdot 2 \cdot 1} = \frac{18!}{3! \cdot 15!}$.
One can prove (by basically appyling the same technique) that this is not only valid for this example, but also for $\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$.